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A SINGULAR BOUNDARY VALUE PROBLEM ARISING IN A PRE-BREAKDOWN GAS DISCHARGE

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A singular boundary value problem arising in a pre-breakdown gas discharge*)

by

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ABSTRACT

We consider the nonlinear two-point boundary value problem $\varepsilon x y'' + (g(x)-y)y' = 0$, $y(0) = 0$, $y(R) = k$, where g is a given concave function. We prove that the problem has a unique solution and we study the limiting behaviour of this solution as $R \to \infty$ and as $\varepsilon \downarrow 0$.

Furthermore, we show how a so-called pre-breakdown discharge in an ionized gas between two electrodes can be described by an equation of this form, and we interpret the results physically.

KEY WORDS & PHRASES: *singularly perturbed nonlinear two-point boundary value problem; pre-breakdown discharge in an ionized gas between two electrodes.*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we study a model of the electric field which arises in the space between two electrodes. The equation in which we shall be engaged is

$$(1.1)\qquad \varepsilon x y'' + (g(x)-y)y' = 0 \qquad 0 < x < R,$$

where the given function g satisfies the following hypotheses

H_g: $g \in C^2(\mathbb{R}_+;\mathbb{R})$; $g(0) = 0$; $g'(x) > 0$ and $g''(x) < 0$ for all $x \geq 0$.

We are interested in solutions of (1.1) which are subject to the boundary conditions

$$(1.2)\qquad y(0) = 0,$$

$$(1.3)\qquad y(R) = k.$$

We assume that the given numbers k and R satisfy $0 < k < g(\infty)$ and $R > x_0$, where x_0 is defined as the (unique) root of the equation $g(x) = k$.

In section 2 we shall derive some a priori estimates and we shall prove the existence of a solution of (1.1) - (1.3). In section 3 we prove that the solution is unique.

The main objective of this paper is the study of the dependence of the solution on the parameters ε and R. In section 3 we prove that the solution is a monotone function of ε and of R. From the physical point of view the interesting regions of the parameters are small ε and large R. In section 4 we analyse the limiting behaviour of the solution when R tends to infinity and ε is fixed. It turns out that the solution converges uniformly in x to a function $\bar{y}$ which satisfies (1.1), (1.2) and the limiting form of (1.3), i.e. $\bar{y}(\infty) = k$, if and only if $\varepsilon \leq g(\infty) - k$. If, on the other hand, this inequality is violated, then the solution converges uniformly on compact subsets to a function $\bar{y}$ which satisfies (1.1), (1.2) and $\bar{y}(\infty) = \max\{g(\infty) - \varepsilon, 0\}$. In particular this implies that $\bar{y}$ is identically zero if $\varepsilon \geq g(\infty)$.

In section 5 we analyse the limiting behaviour of the solution when ε tends to zero and R is fixed. It turns out that the solution converges uniformly in x to the function $\tilde{y}(x) = \min\{g(x),k\}$. Moreover we show that y' converges to $\tilde{y}'$ uniformly for $x \in [0,x_0-\delta] \cup [x_0+\delta,R]$ for arbitrary $\delta > 0$. The behaviour of y' in the transition layer near x_0 will be discussed formally in section 6. Since the limits $\varepsilon \downarrow 0$ and $R \to \infty$ (for $\varepsilon \le g(\infty) - k$) are interchangeable, the two separate limits give a complete picture of the limiting behaviour with respect to both parameters.

Finally, in section 7 we shall indicate how the problem arises in physics and we shall interpret the results physically.

Some partial results have been obtained for the case that g is neither increasing nor concave everywhere. This case will be studied in a forthcoming paper.

The problem (1.1) - (1.3) is, in many respects, similar to a singular boundary value problem of rotating fluids studied by HALLAM & LOPER [6]. We also draw attention to the recent paper of CLÉMENT & EMMERTH [2]. They study the limiting behaviour as $\varepsilon \downarrow 0$ for more general problems by using completely different methods.

2. A PRIORI ESTIMATES AND THE EXISTENCE OF A SOLUTION

In this section we consider the problem (1.1) - (1.3) for fixed values of the parameters ε and R. By a solution we shall mean a function $y \in C^2([0,R];\mathbb{R})$ which satisfies (1.1) - (1.3). We first derive some a priori estimates for a solution and its first two derivatives. Subsequently we prove that a solution actually exists by constructing an upper and lower solution and by verifying the appropriate Nagumo condition.

THEOREM 2.1. *Let* y *be a solution, then for all* $x \in (0,R)$

(i) $0 < y(x) < \min\{g(x),k\}$;

(ii) $0 < y'(x) < g'(0)$;

(iii) $-\dfrac{(g'(0))^2}{\varepsilon} < y''(x) < 0$.

PROOF. Let us first prove that $y'(x) > 0$ for all $x \in (0,R)$. Suppose that $y'(x_1) = 0$ for some $x_1 > 0$, then the standard uniqueness theorem for ordinary differential equations implies that $y(x) = y(x_1)$ for all x. Since this is not compatible with the two boundary conditions we conclude that y' is sign-definite. Invoking the boundary conditions once more, we see that the sign has to be positive.

The positivity of y' implies that $0 < y(x) < k$ for $x \in (0,R)$. Next we shall prove that $y(x) < g(x)$. We begin by observing that this inequality holds for $x \geq x_0$. Suppose there is an interval $[x_1,x_2] \subset [0,x_0]$ such that $y - g$ is strictly positive in the interior of $[x_1,x_2]$ and $y(x_1) - g(x_1) = y(x_2) - g(x_2) = 0$. Then $y'(x_2) \leq g'(x_2) < g'(x_1) \leq y'(x_1)$. On the other hand the equation (1.1) implies that $y''(x) > 0$ for $x \in (x_1,x_2)$ and hence $y'(x_2) = y'(x_1) + \int_{x_1}^{x_2} y''(\xi)d\xi > y'(x_1)$. So our assumption must be false since it leads to a contradiction. Thus, $y(x) \leq g(x)$. Now, let us suppose that $y(x_1) = g(x_1)$ for some $x_1 > 0$, then necessarily $y'(x_1) = g'(x_1)$. However, because $y''(x_1) = 0$ (by (1.1)) and $g''(x_1) < 0$, this would imply that $y(x) > g(x)$ in a right-hand neighbourhood of x_1, which is impossible. Hence the inequality is strict for $x \in (0,R]$, and this completes the proof of (i).

From (i), $y'(x) > 0$ and equation (1.1) we deduce that $y''(x) < 0$ for $x \in (0,R)$. Hence $y'(x) < y'(0) \leq g'(0)$ for $x \in (0,R)$ which completes the proof of (ii).

Finally, we note that H_g implies that $g(x) \leq g'(0)x$ and hence that $y''(x) = (\varepsilon x)^{-1}(y(x) - g(x))y'(x) > -(\varepsilon x)^{-1}g(x)g'(0) \geq -\varepsilon^{-1}(g'(0))^2$. This proves property (iii). □

THEOREM 2.2. *There exists a function* $y \in C^2([0,R];\mathbb{R})$ *which satisfies* (1.1) - (1.3).

PROOF. We define two functions α and β by $\alpha(x) := 0$ and $\beta(x) := g(x)$ for $x \in [0,R]$. Moreover, we define a function f by $f(x,y,y') := (\varepsilon x)^{-1}(y-g(x))y'$. Then $\alpha''(x) = 0 \geq 0 = f(x,\alpha(x),\alpha'(x))$ and $\beta''(x) = g''(x) < 0 = f(x,\beta(x),\beta'(x))$ for $x \in (0,R)$. Hence α and β are, respectively, a lower and an upper solution of (1.1). The existence of a solution now follows from [1, Theorem 1.5.1] if we can show that f satisfies a Nagumo condition with respect to the pair α,β. This amounts to finding a positive continuous function h on $[0,\infty)$ such that $|f(x,y,y')| \leq h(|y'|)$ for all $x \in [0,R]$, $\alpha(x) \leq y \leq \beta(x)$ and $y' \in \mathbb{R}$ and, furthermore, such that

$$\int_{R^{-1}\beta(R)}^{\infty} \frac{s}{h(s)}\, ds > \beta(R),$$

cf. [1, Definition 1.4.1]. The function h defined by $h(s) := \varepsilon^{-1}g'(0)(s+1)$ satisfies all these conditions. □

3. A COMPARISON THEOREM

In order to emphasize that we are going to study the dependence of a solution on the parameters ε and R, we introduce the notation $P(\varepsilon,R)$ for the problem (1.1) - (1.3). The main result of this section is a comparison theorem which is proved by standard maximum principle arguments. As corollaries we obtain that the solution is unique and that it depends in a monotone fashion on both ε and R.

THEOREM 3.1. *Let y_i be a solution of $P(\varepsilon_i,R_i)$ for $i = 1,2$ and suppose that $R_2 \geq R_1 > x_0$ and $\varepsilon_2 \geq \varepsilon_1$. Then $y_1(x) \geq y_2(x)$ for $0 < x < R_1$. Moreover, if one of the inequalities for the parameters is strict, then so is the inequality for the solutions.*

PROOF. Let the function m be defined by $m(x) := y_1(x) - y_2(x)$. Suppose that m achieves a nonpositive minimum on $(0,R_1)$, i.e. suppose that for some $x_1 \in (0,R_1)$, $m(x_1) \leq 0$, $m'(x_1) = 0$ and $m''(x_1) \geq 0$. By subtracting the equation for y_2 from the one for y_1 we obtain

$$\varepsilon_1 x_1 m''(x_1) - (\varepsilon_2-\varepsilon_1)x_1 y_2''(x_1) - y_1'(x_1)m(x_1) = 0.$$

However, all the terms on the left-hand side of this equality are nonnegative and if either $\varepsilon_2 > \varepsilon_1$ or $m(x_1) < 0$ at least one of them is positive. If $\varepsilon_1 = \varepsilon_2$ and $m(x_1) = 0$ then the uniqueness theorem for ordinary differential equations implies that $m(x) = 0$ for all $x \in [0,R_1]$, which cannot be true if $R_2 > R_1$. So we see that m cannot achieve a negative minimum and that m cannot become zero on $(0,R_1)$ if one of the inequalities for the parameters is strict. Since $m(0) = 0$ and $m(R_1) \geq 0$ this proves the theorem. □

COROLLARY 3.2. *The problem* $P(\varepsilon,R)$ *has one and only one solution.*

PROOF. We know that at least one solution exists (Theorem 2.2). Let both y_1 and y_2 satisfy $P(\varepsilon,R)$, then Theorem 3.1 implies that $y_1(x) \geq y_2(x)$ but likewise that $y_2(x) \geq y_1(x)$. Hence, $y_1(x) = y_2(x)$ for $x \in [0,R]$. □

COROLLARY 3.3. *Let* $y = y(x;\varepsilon,R)$ *be the solution of* $P(\varepsilon,R)$. *Then* y *is a monotone decreasing function of* ε *for each* $R > x_0$ *and each* $x \in (0,R)$, *and* y *is a monotone decreasing function of* R *for each* $\varepsilon > 0$ *and each* $x \in (0,R)$.

4. THE LIMITING BEHAVIOUR AS $R \to \infty$

In this section we study the limiting behaviour as $R \to \infty$ of the solution $y = y(x;\varepsilon,R)$ of the problem $P(\varepsilon,R)$. Since y is a bounded and monotone function of R, the definition $\bar{y}(x;\varepsilon) := \lim_{R\to\infty} y(x;\varepsilon,R)$ makes sense for all $x,\varepsilon > 0$. This definition implies at once that $\bar{y}(0;\varepsilon) = 0$ and that $\bar{y}$ is a nondecreasing function of x and a nonincreasing function of ε.

From the estimates in Theorem 2.1 we obtain, via the Arzela-Ascoli theorem, that both $y(\cdot;\varepsilon,R)$ and $y'(\cdot;\varepsilon,R)$ converge uniformly on compact subsets. Invoking equation (1.1) we see that the same must be true for $y''(\cdot;\varepsilon,R)$. It follows that $\bar{y}(\cdot;\varepsilon)$ belongs to $C^2(\mathbb{R}_+;\mathbb{R})$ and satisfies equation (1.1). Now it remains to determine $\bar{y}(\infty;\varepsilon)$. We will estimate $\bar{y}(\infty;\varepsilon)$ from below by constructing a more subtle lower solution for y. But first we prove a result which can be used to estimate $\bar{y}(\infty;\varepsilon)$ from above.

LEMMA 4.1. *Let* $z \in C^2(\mathbb{R}_+;\mathbb{R})$ *satisfy equation* (1.1) *and* $z(0) = 0$. *Suppose that* $z(\infty) := \lim_{x\to\infty} z(x)$ *exists and satisfies* $0 < z(\infty) < \infty$. *Then* $z(\infty) \leq g(\infty) - \varepsilon$.

PROOF. Both z and z' are positive on $(0,\infty)$ (cf. the proof of Theorem 2.1). For the purpose of contradiction, let us suppose that $z(\infty) > g(\infty) - \varepsilon$. Let x_1 be such that $\beta := \varepsilon^{-1}(z(x_1) - g(\infty)) > -1$. Then $z(x) - g(x) \geq z(x_1) - g(\infty) = \varepsilon\beta$ for all $x \geq x_1$. Integrating equation (1.1) twice from x_1 to x we obtain

$$z(x) = z(x_1) + z'(x_1) \int_{x_1}^{x} \exp\left(\int_{x_1}^{\xi} \frac{z(\eta) - g(\eta)}{\varepsilon\eta}\, d\eta\right) d\xi.$$

Thus, for $x \geq x_1$,

$$z(x) \geq z'(x_1) \int_{x_1}^{x} \exp\left(\beta \ln \frac{\xi}{x_1}\right) d\xi = \frac{x_1 z'(x_1)}{\beta + 1} \left(\left(\frac{x}{x_1}\right)^{\beta+1} - 1\right).$$

Since $\beta+1 > 0$ this would imply that $z(x) \to \infty$ as $x \to \infty$. Hence the assumption that $z(\infty) > g(\infty) - \varepsilon$ must be false. □

We define a function $s = s(x;\lambda,x_1,\nu)$ by

$$s(x;\lambda,x_1,\nu) := \lambda\left(1 - \left(\frac{x}{x_1}\right)^{-\nu}\right) \tag{4.1}$$

and we investigate which conditions for the parameters λ, x_1 and ν guarantee that $s'' \geq f(x,s,s')$ for $x \geq x_1$ (recall that $f(x,y,y') = (\varepsilon x)^{-1}(y-g(x))y'$). A simple computation shows that this inequality holds indeed for all $x \geq x_1$ if and only if $g(x_1) - \lambda - \varepsilon\nu - \varepsilon \geq 0$, or equivalently, $\nu \leq \varepsilon^{-1}(g(x_1) - \lambda) - 1$. The latter inequality can be satisfied for some *positive* value of ν if and only if $\lambda < g(x_1) - \varepsilon$. In its turn this inequality can be satisfied for sufficiently large x_1 and some *positive* value of λ if and only if $g(\infty)-\varepsilon > 0$.

We now have all the ingredients at hand to prove the following theorem.

THEOREM 4.2.

(i) *If* $\varepsilon \leq g(\infty) - k$ *then* $\bar{y}(\infty;\varepsilon) = k$ *and* $\lim_{R\to\infty} \sup_{0\leq x\leq R} |y(x;\varepsilon,R) - \bar{y}(x;\varepsilon)| = 0$;

(ii) *if* $g(\infty) - k < \varepsilon < g(\infty)$ *then* $\bar{y}(\infty;\varepsilon) = g(\infty) - \varepsilon$;

(iii) *if* $\varepsilon \geq g(\infty)$ *then* $\bar{y}(x;\varepsilon) = 0$ *for all* $x \geq 0$.

PROOF. (i) For any $\lambda < k$ we can choose x_1 such that $\lambda < g(x_1) - \varepsilon$ and subsequently ν such that $0 < \nu \leq \varepsilon^{-1}(g(x_1) - \lambda) - 1$. For these values of the parameters, s is a lower solution on the interval $[x_1,R]$. The function t defined by $t(x) := k$ is an upper solution and f satisfies a Nagumo condition with respect to the pair s,t and the interval $[x_1,R]$. It follows that the inequality

$$s(x;\lambda,x_1,\nu) \leq y(x;\varepsilon,R) \leq k,$$

which holds for $x = x_1$ and for $x = R$, actually is satisfied for all

$x \in [x_1,R]$. By taking first the limit $R \to \infty$ and then the limit $x \to \infty$ we obtain

$$\lambda \leq \bar{y}(\infty;\varepsilon) \leq k.$$

Since this inequality holds for $\lambda < k$, necessarily $\bar{y}(\infty;\varepsilon) = k$. This result and the monotonicity of y with respect to x together imply that the convergence of y to $\bar{y}$ is in fact uniform in x (we refer to [3, Lemma 2.4] for the proof of this statement).

(ii). If $g(\infty) - k \leqslant \varepsilon < g(\infty)$, we can make s into a lower solution by a suitable choice of x_1 and ν if and only if $\lambda < g(\infty) - \varepsilon$. The argument we used in the proof of (i) now shows that $\bar{y}(\infty;\varepsilon) \geq g(\infty) - \varepsilon$. On the other hand, Lemma 4.1 implies that $\bar{y}(\infty;\varepsilon) \leq g(\infty) - \varepsilon$. So $\bar{y}(\infty;\varepsilon) = g(\infty) - \varepsilon$.

(iii) From Lemma 4.1 we deduce that no solution of (1.1) with a positive limit at infinity can exist if $\varepsilon \geq g(\infty)$. Hence $\bar{y}(\infty;\varepsilon) = 0$ and consequently $\bar{y}(x;\varepsilon) = 0$ for all $x \geq 0$. □

The results of this section are at the same time results concerning the existence and non-existence of a solution of the problem $P(\varepsilon,\infty)$ defined by (1.1), (1.2) and $\lim_{x\to\infty} y(x) = k$. By exactly the same arguments which we used before one can derive the bounds of Theorem 2.1 and one can show that there exists at most one solution of $P(\varepsilon,\infty)$. For convenience we formulate this result in the following theorem.

THEOREM 4.3. *There exists a function* $y \in C^2(\mathbb{R}_+;\mathbb{R})$ *which satisfies* (1.1), (1.2) *and the condition* $\lim_{x\to\infty} y(x) = k$ *if and only if* $\varepsilon \leq g(\infty) - k$. *If it exists, it is unique and it satisfies the inequalities given in Theorem* 2.1.

5. THE LIMITING BEHAVIOUR AS $\varepsilon \downarrow 0$

Throughout this section $R > x_0$ will be fixed and we will suppress the dependence on R in the notation, because it is inessential. The solution y of (1.1) - (1.3) is a bounded and monotone function of ε and we define $\tilde{y}(x) := \lim_{\varepsilon\downarrow 0} y(x;\varepsilon)$. From Theorem 2.1(i) and (ii) and the Arzela-Ascoli theorem we deduce that $\tilde{y}$ is continuous and that in fact $\lim_{\varepsilon\downarrow 0} \sup_{0\leq x\leq R} |\tilde{y}(x) - y(x;\varepsilon)| = 0$.

THEOREM 5.1. $\tilde{y}(x) = \min\{g(x),k\}$.

PROOF. From Theorem 2.1(i) we know that $\tilde{y}(x) \leq \min\{g(x),k\}$. Take any $x < x_0$, then $\tilde{y}(x) < k$. We claim that this implies that $\liminf_{\varepsilon\downarrow 0} y'(x;\varepsilon) > 0$. Indeed, suppose that the sequence $\{\varepsilon_i\}$ is such that $\varepsilon_i \downarrow 0$ and $y'(x;\varepsilon_i) \downarrow 0$ as $i \to \infty$, then by taking the limit $i \to \infty$ in the relation

$$k = y(R;\varepsilon_i)$$
$$= y(x;\varepsilon_i) + \int_x^R y'(\xi;\varepsilon_i)d\xi \leq y(x;\varepsilon_i) + (R-x)y'(x;\varepsilon_i),$$

we arrive at the conclusion that $\tilde{y}(x) \geq k$, which is impossible.

Integrating equation (1.1) from 0 to x we obtain

$$\varepsilon(y'(x;\varepsilon) - y'(0;\varepsilon)) = \int_0^x \frac{y(\xi;\varepsilon) - g(\xi)}{\xi}\, y'(\xi;\varepsilon)d\xi. \tag{5.1}$$

Suppose that $x < x_0$ and $\max_{0\leq\xi\leq x} |\tilde{y}(\xi) - g(\xi)| > 0$ then, since $g'(0) > y'(\xi;\varepsilon) \geq y'(x;\varepsilon)$ for $0 < \xi \leq x$ and $\liminf_{\varepsilon\downarrow 0} y'(x;\varepsilon) > 0$, the right-hand side of (5.1) is bounded away from zero as $\varepsilon \downarrow 0$. However, this is impossible since the left-hand side tends to zero as $\varepsilon \downarrow 0$. So $\tilde{y}(x) = g(x)$ for all $x < x_0$, and by continuity $\tilde{y}(x_0) = k$. The function $\tilde{y}$, being the limit of monotone functions, is monotone nondecreasing. Hence $\tilde{y}(x) \geq k$ for $x > x_0$ and consequently $\tilde{y}(x) = k$ for $x > x_0$. □

By taking $\varepsilon = 0$ in (1.1) we obtain the reduced equation

$$(g(x) - y)y' = 0. \tag{5.2}$$

The limiting function $\tilde{y}$ satisfies the boundary conditions (1.2) and (1.3) and the equation (5.2) except at the point $x = x_0$, where $\tilde{y}'$ is not defined. Motivated in part by the physical application (cf. section 7) we shall now investigate the limiting behaviour of $y'(x;\varepsilon)$ as $\varepsilon \downarrow 0$. It will then become even more apparent that $x = x_0$ is an exceptional point. The following lemma is needed in the proof of Theorem 5.3, but it is of some interest in itself.

LEMMA 5.2. *Let* $\delta > 0$ *be arbitrary. For any* $\varepsilon_0 > 0$ *there exists an* $M > 0$ *such that* $0 < g(x) - y(x;\varepsilon) < M\varepsilon x$ *for all* $x \in [0,x_0-\delta]$ *and all* $\varepsilon \in (0,\varepsilon_0)$.

PROOF. Let $\delta > 0$ and $\varepsilon_0 > 0$ arbitrary. We define

$$m(\varepsilon) := \min_{x_0-\delta \le x \le x_0-\frac{1}{2}\delta} \{g(x) - y(x;\varepsilon)\}.$$

Then there exist positive constants C_i, $i = 1,2,3$, such that for $\varepsilon \in (0,\varepsilon_0)$

$$m(\varepsilon) \le C_1 \int_{x_0-\delta}^{x_0-\delta/2} (g(\xi) - y(\xi;\varepsilon))d\xi$$

$$\le C_2 \int_{x_0-\delta}^{x_0-\delta/2} \frac{g(\xi) - y(\xi;\varepsilon)}{\xi} y'(\xi;\varepsilon)d\xi \le C_3\varepsilon$$

(see the proof of Theorem 5.1 and in particular formula (5.1)). Let the function $v = v(x;\varepsilon)$ be defined by $v(x;\varepsilon) := g(x) - y(x;\varepsilon) - M\varepsilon x$, where the constant $M > 0$ is still at our disposal. Then v satisfies the equation

$$\varepsilon x v'' - y'(x;\varepsilon)v = \varepsilon x(g''(x) + My'(x;\varepsilon))$$

and consequently $\varepsilon x v'' - \mu v > 0$ if $M > \gamma\mu^{-1}$, $\varepsilon \in (0,\varepsilon_0)$ and $x \in (0,x_0 - \frac{1}{2}\delta]$, where the positive numbers γ and μ are defined by

$$\gamma := - \inf_{0<x\le x_0-\frac{1}{2}\delta} g''(x)$$

and

$$\mu := \inf_{0<\varepsilon<\varepsilon_0} y'(x_0 - \frac{\delta}{2};\varepsilon).$$

So if $M > \gamma\mu^{-1}$ and $\varepsilon \in (0,\varepsilon_0)$, then v cannot assume a nonnegative maximum on $(0,x_0-\frac{1}{2}\delta)$. Let $x(\varepsilon)$ be such that $g(x) - y(x;\varepsilon)$ achieves its minimum on the set $[x_0-\delta,x_0-\frac{1}{2}\delta]$ in the point $x = x(\varepsilon)$. Then $v(x(\varepsilon);\varepsilon) = m(\varepsilon) - M\varepsilon x(\varepsilon) < 0$ if $M > (x_0-\delta)^{-1} C_3$. Since $v(0;\varepsilon) = 0$, this implies that for $M > \max\{\gamma\mu^{-1},(x_0-\delta)^{-1} C_3\}$ $v(x;\varepsilon) < 0$ for $x \in (0,x(\varepsilon))$ and a fortiori for $x \in (0,x_0-\delta)$. □

THEOREM 5.3. *Let $\delta > 0$ be arbitrary. Then*

(i) $\lim_{\varepsilon\downarrow 0} \sup_{0\le x\le x_0-\delta} |g'(x) - y'(x;\varepsilon)| = 0;$

(ii) $\lim_{\varepsilon\downarrow 0} \sup_{x_0+\delta\le x\le R} |y'(x;\varepsilon)| = 0.$

PROOF. (i) From the equation (1.1), Theorem 2.1(ii) and Lemma 5.2 we deduce that $-g'(0)M < y''(x;\varepsilon) < 0$ for $x \in [0,x_0-\delta]$ and $\varepsilon \in (0,\varepsilon_0)$. By the Arzela-Ascoli theorem this implies that the limit set of $\{y'(\cdot;\varepsilon) \mid \varepsilon > 0\}$ as $\varepsilon \downarrow 0$ is nonempty in $C([0,x_0-\delta];\mathbb{R})$. The result now follows from the fact that y tends to g on $[0,x_0-\delta]$ as $\varepsilon \downarrow 0$.

(ii). Integrating equation (1.1) from $x_0 + \frac{1}{2}\delta$ to x we obtain

$$\varepsilon(y'(x;\varepsilon) - y'(x_0 + \tfrac{1}{2}\delta;\varepsilon)) = \int_{x_0+\frac{1}{2}\delta}^{x} \frac{y(\xi;\varepsilon) - g(\xi)}{\xi}\, y'(\xi;\varepsilon)d\xi.$$

For $x \in [x_0+\delta,R]$ the right-hand side is smaller than $\frac{1}{2}\delta R^{-1}(k-g(x_0+\frac{\delta}{2}))y'(x;\varepsilon)$. Consequently $0 < y'(x;\varepsilon) < 2g'(0)\varepsilon R\delta^{-1}(g(x_0+\frac{\delta}{2}) - k)^{-1}$. □

In the next section we shall concentrate on a formal approximation for y' in the neighbourhood of $x = x_0$.

In section 4 it was shown that the problem $P(\varepsilon,\infty)$ has a unique solution for ε sufficiently small. The analysis of this section can be repeated, mutatis mutandis, to derive the analogous results concerning the limiting behaviour of this solution as $\varepsilon \downarrow 0$. In particular this implies that the limits $\varepsilon \downarrow 0$ and $R \to \infty$ are interchangeable.

6. THE TRANSITION LAYER

In Theorem 5.3 we have shown that y' converges nonuniformly on the interval $[0,R]$ as $\varepsilon \downarrow 0$. This feature is typical for a singular perturbation problem. In this section we use the standard method of the stretching of a variable to obtain more information about the behaviour of y' near the transition point $x = x_0$.

Our starting point is the equation

$$(6.1)\qquad \varepsilon x z'' + \varepsilon z' - \varepsilon x \frac{(z')^2}{z} + (g'(x) - z)z = 0,$$

which is obtained by differentiation of (1.1), followed by elimination of y and the substitution $y' = z$. By the stretching of x near x_0 we mean the introduction of a local coordinate ξ according to $x = x_0 + \xi\varepsilon^{\alpha}$. If we make this substitution in (6.1) and if we subsequently retain only the terms of lowest order in ε, then it depends on the value of α what the resulting equation will be. One easily verifies that the choice $\alpha = \frac{1}{2}$ leads to a significant equation, namely to

$$(6.2)\qquad x_0 z'' - x_0 \frac{(z')^2}{z} + (g'(x_0) - z)z = 0.$$

To this equation we add the condition that its solution should match the limits of y' to the left and to the right of x_0, respectively. This amounts to the condition

$$(6.3)\qquad \lim_{\xi\to-\infty} z(\xi) = g'(x_0), \quad \lim_{\xi\to+\infty} z(\xi) = 0.$$

We rewrite (6.2) as a two-dimensional autonomous first order system

$$(6.4)\qquad \begin{aligned} z' &= u, \\ u' &= \frac{u^2}{z} + \frac{1}{x_0}(z - g'(x_0))z, \end{aligned}$$

and we analyse the trajectories in the (z,u)-phase plane. If we linearize the system about the equilibrium point $(g'(x_0),0)$ we find the corresponding eigenvalues and eigenvectors to be

$$\lambda_{\pm} = \pm\sqrt{\frac{g'(x_0)}{x_0}}$$

and

$$(1,\lambda_{\pm})^T.$$

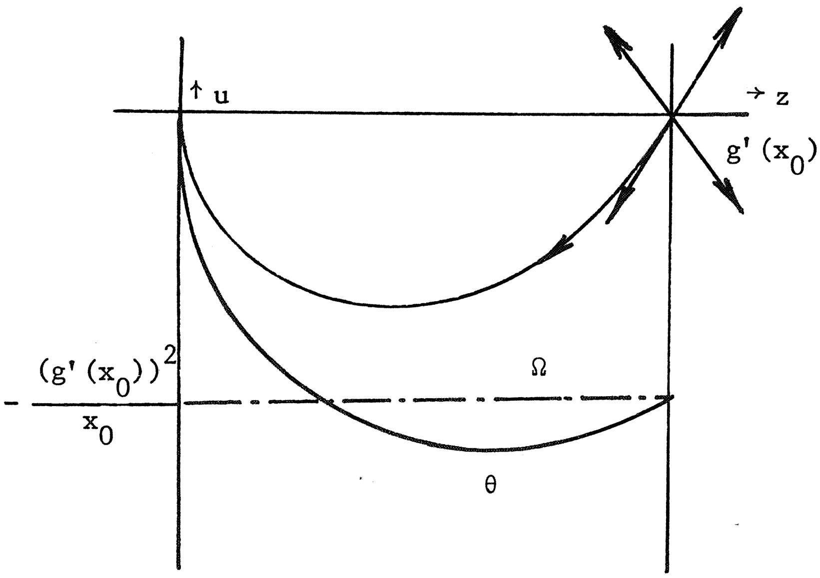
So $(g'(x_0),0)$ is a saddle point and the unstable manifold is tangent to the vector $(1,\lambda_+)^T$.

Our next objective is to overcome the difficulty that the right-hand side of (6.4) is singular for $z = 0$. Consider the set Ω enclosed by the z-axis, the line $z = g'(x_0)$ and the arc θ which is defined by the parameterization $(z,u) = (\theta_1(t),\theta_2(t))$, $1 \le t < \infty$, with

$$\theta_1(t) := g'(x_0)\exp\left\{\frac{g'(x_0)}{2x_0}(1-t^2)\right\}$$

and

$$\theta_2(t) := -\frac{(g'(x_0))^2}{x_0}\,t\exp\left\{\frac{g'(x_0)}{2x_0}(1-t^2)\right\}.$$



Along $\partial\Omega$ the vector field defined by the right-hand side of (6.4) is always pointing inward. For the part of $\partial\Omega$ where $u = 0$ or $z = g'(x_0)$ this is immediately clear from (6.4), and along θ it follows from the fact that the outcome of a calculation of the inner product of the vector field with $(\dot\theta_2,-\dot\theta_1)^T$, i.e. with the inward pointing normal to θ, is

$$\frac{(g'(x_0))^4}{x_0^2}\,t\exp\left\{\frac{3g'(x_0)}{2x_0}(1-t^2)\right\},$$

which is positive for $1 \le t < \infty$.

In Ω the vector field is continuous if we define its value at (0,0) to be zero. The unstable manifold of the saddle point which enters Ω can be continued and it cannot leave Ω through $\partial\Omega\backslash(0,0)$. Hence it enters the singular point (0,0). So we have shown that the problem (6.2) - (6.3) has (modulo translation) a unique solution.

Equation (6.2) can be solved explicitly for ξ as a function of z. If we require that $\lim_{\xi\to-\infty} z(\xi) = g'(x_0)$ and $z(0) = z_0$ we find that

$$\xi = \sqrt{\frac{x_0}{2g'(x_0)}} \int_{\ln \frac{z_0}{g'(x_0)}}^{\ln \frac{z}{g'(x_0)}} \frac{du}{\sqrt{e^u-1-u}}.$$

From this expression we deduce that

$$z(\xi) \sim \exp\left(-\frac{g'(x_0)}{2x_0}\xi^2\right) \qquad \text{for } \xi \to +\infty.$$

Let z be a solution of (6.2) - (6.3). The idea of singular perturbation theory is that z describes the shape of y' near $x = x_0$ for small values of ε, and that one can approximate y' uniformly on $[0,R]$ by using the building-stones z and $\tilde{y}'$. We shall not give a precise formulation of this idea. The interested reader is referred to a paper of FIFE [4] where similar problems are treated and rigorous proofs are given.

7. APPLICATION TO A THEORETICAL MODEL OF AN IONIZED GAS BETWEEN TWO ELECTRODES

7.1. The physical background

MARODE et al. [8] consider an ionized gas between two electrodes in which the ions can be represented by a density $n_i(\vec{r})$ and the electrons by a density $n_e(\vec{r})$. Due to the heavy weight of the ions, the density $n_i(\vec{r})$ may be considered as fixed, whereas the highly mobile electrons assume a spatial distribution in thermal equilibrium with the given $n_i(\vec{r})$. The problem is then to find $n_e(\vec{r})$ for given $n_i(\vec{r})$. A special situation of practical interest is considered, namely a pre-breakdown discharge which spreads out in filamentary form, as discussed by GALLIMBERTI [5] and MARODE [7]. For this case, where there is cylindrical symmetry, MARODE et al. [8] have derived the following equation:

$$(7.1) \qquad -\tilde{\varepsilon}\,\frac{1}{r}\,\frac{d}{dr}\left(\frac{r}{n_e(r)}\,\frac{dn_e(r)}{dr}\right) + n_e(r) = n_i(r),$$

where $\tilde{\varepsilon}$ is a small positive constant proportional to the temperature. In addition, $n_e(r)$ has to satisfy the boundary condition

$$(7.2) \qquad \frac{dn_e}{dr}(0) = 0$$

and the condition

$$(7.3) \qquad \int_0^\infty (n_i(r) - n_e(r))\,r\,dr = N > 0,$$

where N represents the excess of ions.

The electron density $n_e(r)$ is to be determined from equations (7.1) - (7.3). We assume that $n_i(r)$ is a smooth decreasing function of r. An explicit choice of MARODE et al. [8] is $n_i(r) = e^{-r^2}$, because the ions are supposed to have a Gaussian distribution due to diffusion.

7.2. A transformation of the equation

Let us make the following change of function:

$$(7.4) \qquad y(r^2) = \int_0^r n_e(\rho)\rho\,d\rho.$$

Obviously y represents the total quantity of electrons per unit length in a cylinder of radius r. We multiply (7.1) by r and integrate from 0 to r to find

$$-\tilde{\varepsilon}\,\frac{r}{n_e(r)}\,\frac{dn_e(r)}{dr} + \int_0^r n_e(\rho)\rho\,d\rho = \int_0^r n_i(\rho)\rho\,d\rho,$$

whence upon putting $x = r^2$

$$-2\tilde{\varepsilon}xy''(x) + y(x)y'(x) = \left(\int_0^{\sqrt{x}} n_e(\rho)\rho d\rho\right)y'(x).$$

We finally obtain the equation (1.1), the boundary condition (1.2) and the limiting form of the boundary condition (1.3), i.e. $y(\infty) = k$, if we put

$$\varepsilon = 2\tilde{\varepsilon}, \quad g(x) = \int_0^{\sqrt{x}} n_i(\rho)\rho d\rho, \quad \text{and } k = \int_0^{\infty} n_i(\rho)\rho d\rho - N.$$

Since there are fewer electrons than ions, we have $k < g(\infty)$, as assumed in the paper.

Problems (7.1) - (7.3) and $P(\varepsilon,\infty)$ are equivalent under the change of function (7.4) and the change of variable $x = r^2$. (Since $y' > 0$, $n_e(r) = 2y'(r^2)$ is well defined.) In this paper, starting from the study of the corresponding problem on a finite interval, we have established the main characteristics of $P(\varepsilon,\infty)$.

7.3. Physical interpretation of the results

Given the properties for y and y' which were proved in the previous sections, we deduce in particular the following properties for n_e:

(a) for $\tilde{\varepsilon} \leq N/2$, there exists a unique solution of problem (7.1) - (7.3) and there does not exist any otherwise.

(b) Let $r_0 > 0$ be defined by $\int_{r_0}^{\infty} n_i(\rho)\rho d\rho = N$. Then for any $\delta \in (0,r_0)$

$$\lim_{\tilde{\varepsilon}\downarrow 0} \sup_{r\in[0,r_0-\delta]} |n_e(r) - n_i(r)| = 0,$$

$$\lim_{\tilde{\varepsilon}\downarrow 0} \sup_{r\in[r_0+\delta,\infty)} |n_e(r)| = 0.$$

(c) In the neighbourhood of r_0, n_e exhibits a transition layer behaviour.

The most interesting physical result is that at temperatures such that $\tilde{\varepsilon} << 1$, the density of electrons is not smoothly distributed in the ionized region. It is, on the contrary, almost confined to a cylindrical neighbourhood of the discharge axis where, as a consequence, the plasma is quasi-neutral. At a certain distance from the axis a transition occurs to a region of positive space charge. The width of the transition layer is of order $\tilde{\varepsilon}^{\frac{1}{2}}$, which MARODE et al. [8] show to have the form of a Debye shielding length, resulting from a balance between thermal and Coulomb energy.

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